

STRONGLY EXTENSIONAL HOMOMORPHISM OF IMPLICATIVE SEMIGROUPS WITH APARTNESS

DANIEL ABRAHAM ROMANO

ABSTRACT. The setting of this research is the Bishop's constructive mathematics. Following the ideas of Chan and Shum, exposed in their famous paper "Homomorphisms of implicative semigroups" ([10]), we discuss the homomorphisms between the implicative semigroups S and T with apartness as a continuation of our recently published article [24]. The specificity of this research is the application of Intuitionistic logic instead of Classical. In addition, we concentrate on the structure of the implicative semigroups with apartness and then on their interaction which do not appear in the classical case ([10]). In this paper, the concept of ordered anti-filter has been associated with strongly extensional homomorphisms between such semigroup.

1. INTRODUCTION

The notions of implicative semigroup and ordered filter were introduced by Chan and Shum [10]. For the first generalization of implicative semilattice see Nemitz [18] and Blyth [7]. Moreover, there exists a close relationship between implicative semigroups and other domains. For example, there is a lot of implications in mathematical logic and set theory (see Birkhoff [6]). For the general development of implicative semilattice theory, the ordered filters play an important role. It has been shown by Nemitz [18]. Motivated by this, Chan and Shum [10] established some elementary properties and constructed quotient structure of implicative semigroups via ordered filters. Jun [13, 14], Jun, Meng and Xin [15], Jun and Kim [16] and Lee, Shum and Wu [25, 26] discussed ordered filters of implicative semigroups. Bang and So [1] analyzed some special substructures in implicative semigroups.

In paper [24], in setting of Bishop's constructive mathematics, following the ideas of Chan and Shum and other authors mentioned above, we introduced the

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notion of implicative semigroups with (tight) apartness and gave some fundamental characterization of these semigroups. In [24] and in this article, we use sets with apartness and anti-order relations introduced by the author, instead of partial order. See for example [11, 20, 21]. In this case, it is an excise relation, researched by Barony [2], Greenleaf [12], Negri [19] and von Plato [28]). So, in this research, we study side effects induced by existence of apartness and anti-orders. Additionally, in [24] we introduced the notion of anti-filter in an implicative semigroup and described its connections with filter.

In third part of this article, we discuss homomorphism between implicative semigroups. In Theorem 3.1 we give some fundamental characteristics of homomorphisms between implicative semigroups. For example:

- strongly extensional homomorphism between implicative semigroups is a reverse isotone map;
- the pre-image of the strongly complement of $\{1\}$ is an ordered anti-filter.

At the end of this research we give a theorem about factorization of this homomorphism (Theorem 3.3).

2. PRELIMINARIES

2.1. Set with apartness. This investigation is in Bishop's constructive algebra in a sense of papers [11, 20, 21, 22, 21] and books [3, 4, 5, 8, 9], [27](Chapter 8: Algebra). Let $(S, =, \neq)$ be a constructive set (i.e. it is a relational system with the relation " \neq "). The diversity relation " \neq " ([4]) is a binary relation on S , which satisfies the following properties:

$$\neg(x \neq x), x \neq y \implies y \neq x, x \neq y \wedge y = z \implies x \neq z .$$

If it satisfies the following condition

$$(\forall x, z \in S)(x \neq z \implies (\forall y \in S)(x \neq y \vee y \neq z)),$$

then, it is called apartness (A. Heyting). In this paper, we assume that the basic apartness is tight, i.e. it satisfies the following

$$(\forall x, y \in S)(\neg(x \neq y) \implies x = y).$$

For subset X of S , we say that it is a strongly extensional subset of S if and only if $(\forall x \in X)(\forall y \in S)(x \neq y \vee y \in S)$. Following Bridges and Vita's (see for example [9]) definition for subsets X and Y of S , we say that set X is set-set apartness from Y , and it is denoted by $X \bowtie Y$, if and only if $(\forall x \in X)(\forall y \in Y)(x \neq y)$. We set $x \bowtie Y$, instead of $\{x\} \bowtie Y$, and, of course, $x \neq y$ instead of $\{x\} \bowtie \{y\}$. With $X^C = \{x \in S : x \bowtie X\}$ we denote apartness complement of X . For a function

$f : (S, =, \neq) \longrightarrow (T, =, \neq)$ we say that it is a strongly extensional if and only if

$$(\forall a, b \in S)(f(a) \neq f(b) \implies a \neq b).$$

For relation $\alpha \subseteq S \times S$, we say that it is an anti-order relation on semigroup S , if it is consistent, cotransitive and linear

$$\alpha \subseteq \neq, \alpha \subseteq \alpha * \alpha, \neq \subseteq \alpha \cup \alpha^{-1},$$

where α has to be compatible with the semigroup operation in the following way

$$(\forall x, y, z \in S)((xz, yz) \in \alpha \vee (zx, zy) \in \alpha) \implies (x, y) \in \alpha.$$

Here, $*$ is the filed product between relations defined by the following way: If α and β are relations on set S , then filed product $\beta * \alpha$ of relation α and β is the relation given by $\{(x, z) \in X \times X : (\forall y \in X)((x, y) \in \alpha \vee (y, z) \in \beta)\}$.

For undefined notions and notations the reader is referred to their following papers [20], [21], [22], [23] and [24].

2.2. Implicative semigroups with apartness. We recall some definitions and results. By a *negatively anti-ordered* semigroup (briefly, n.a-o. semigroup) we mean a set S with an anti-order α and a binary operation $'\cdot'$ (we will write xy instead $x \cdot y$) such that for all $x, y, z \in S$ we have to have:

- (1) $(xy)z = x(yz)$,
- (2) $(xz, yz) \in \alpha$ or $(zx, zy) \in \alpha$ implies $(x, y) \in \alpha$, and
- (3) $(xy, x) \bowtie \alpha$ and $(xy, y) \bowtie \alpha$.

In that case for anti-order α we will say that it is a *negative anti-order relation* on semigroup. The operation $'\cdot'$ is extensional and strongly extensional function from $S \times S$ into S , i.e. it has to be

$$\begin{aligned} (x, y) = (x', y') &\implies xy = x'y' \\ (xy \neq x'y \vee yx \neq yx') &\implies x \neq x' \end{aligned}$$

for any elements x, x', y, y' of S .

A n.a-o. semigroup $(S, =, \neq, \cdot, \alpha)$ is said to be *implicative* if there is an additional binary operation $\otimes : S \times S \longrightarrow S$ such that the following is true

- (4) $(z, x \otimes y) \in \alpha \iff (zx, y) \in \alpha$ for any elements x, y, z of S .

In addition, let us recall that the internal binary operation $'\otimes'$ must satisfy the following implications:

$$\begin{aligned} (a, b) = (u, v) &\implies a \otimes b = u \otimes v, \\ a \otimes b \neq u \otimes v &\implies (a, b) \neq (u, v). \end{aligned}$$

The operation \otimes is called *implication*. From now on, an implicative n.a-o. semigroup is simply called an *implicative semigroup*.

In any implicative semigroup S there exist a special element of S , the biggest element in (S, α^C) , which is the left neutral element in (S, \cdot) .

In this section we will begin with standard definition (Chan and Shum [[10], Definition 2.1]) of ordered filter. Let S be an implicative semigroup and let F be a nonempty subset of S . Then F is called an *ordered filter* of S if

- (F1) $xy \in F$ for every $x, y \in F$, that is, F is a subsemigroup of S , and
(F2) If $x \in F$ and $(x, y) \bowtie \alpha$, then $y \in F$.

As we have seen that the condition (F1) is a condition for a subset F of a semigroup S , until the condition (F2) supplies that F is an upper set. As it is usual in the Constructive mathematics, we can introduce a special (inhabited) proper subset of implicative semigroup S claiming that subset G of S satisfies the following conditions:

- (G1) $xy \in G \implies x \in G \vee y \in G$, that is, G is a cosubsemigroup of S and
(G2) $y \in G \implies (x, y) \in \alpha \vee x \in G$.

This subset of S is called *anti-filter*. We can easily verify that the anti-filter is a strongly extensional subset of S . Moreover, strong compliment G^C of an anti-filter G is a filter in S .

3. THE MAIN RESULTS

Let $S = (S, =, \neq, \cdot, \alpha, \otimes)$ and $T = (T, =, \neq, \cdot, \beta, \otimes)$ be two implicative semigroups and let $f : S \rightarrow T$ be a strongly extensional mapping from S in T . As the usual procedure in the construction of a mathematical system, for mapping f we say that it is a homomorphism between implicative semigroups S and T if

$$(\forall x, y \in S)(f(x \otimes y) = f(x) \otimes f(y))$$

holds.

The homomorphisms between the implicative semigroups have been studied by Chan and Shum in [10]. In this section, our aim is to extend the results in Chan and Shum [10] to implicative semigroups with apartness and strongly extensional homomorphisms. We first notice that the following implication holds

$$f(x) \otimes f(y) \neq f(x') \otimes f(y) \implies f(x) \neq f(x').$$

In fact, for elements x, x', y of S we have the following sequence:

$$\begin{aligned} f(x) \otimes f(y) \neq f(x') \otimes f(y) &\iff f(x \otimes y) \neq f(x' \otimes y) \\ &\iff (f(x) \otimes f(y), f(x') \otimes f(y)) \in \beta \cup \beta^{-1} \\ &\implies (f(x), f(x')) \in \beta \cup \beta^{-1} \\ &\iff f(x) \neq f(x'). \end{aligned}$$

The implication $f(y) \otimes f(x) \neq f(y) \otimes f(x') \implies f(x) \neq f(x')$ follows analogously. Therefore, the homomorphism f is compatible with the operation ' \otimes '.

We now continue to study the implicative semigroups with extensional homomorphisms. It is noted that Theorem 2.2 [10] is a crucial result of this paper because we have to refer this theorem in proving the following Theorem 3.1.

Theorem 3.1. *Let $f : S \longrightarrow T$ be an implicative homomorphism between implicative semigroups S and T . Then the following hold:*

- (1) $f(1) = 1$;
- (2) $f(x) \neq 1 \implies x \neq 1$ for any $x \in S$;
- (3) f is a reverse isotone mapping, that is, if $(f(x), f(y)) \in \beta$ then $(x, y) \in \alpha$;
- (4) If f is surjective, then f is a semigroup homomorphism, that is $f(xy) = f(x)f(y)$;
- (5) $G = f^{-1}(\{1\}^C)$ is an ordered anti-filter in S and valid $G \subseteq \{1\}^C$;
- (6) f is an embedding homomorphism if and only if $G = \{1\}^C$.

Proof. (1) By Theorem 3.3 in [24], we have $f(1) = f(1 \otimes 1) = f(1) \otimes f(1) = 1$.

(2) This assertion immediately follows from (1) and from the fact that f is a strongly extensional homomorphism. Indeed, from $f(x) \neq 1 = f(1)$ follows $x \neq 1$.

(3) Suppose that for $x, y \in S$ hold $(f(x), f(y)) \in \beta$. Then by Theorem 3.4 in [24] it follows that $f(x) \otimes f(y) \neq 1$. Since f is a homomorphism of implicative semigroups, then we have $f(x \otimes y) \neq 1$. Hence, by (2) we have $x \otimes y \neq 1$. Thus, again by Theorem 3.4 in [24], it follows $(x, y) \in \alpha$. So, f is a reverse isotone homomorphism.

(4) We first show that $(f(xy), f(x)f(y)) \in \beta$. As f is onto, there exists an element $z \in S$ such that $f(z) = f(x)f(y)$. Since f is a homomorphism, we have $f(xy) \otimes f(z) = f((xy) \otimes z) = f(x \otimes (y \otimes z)) = f(x) \otimes f(y \otimes z) = f(x) \otimes (f(y) \otimes f(z)) = (f(x)f(y)) \otimes f(z) = 1$ by Theorem 6.2, point (2), in [24]. Thus, by above mentioned Theorem 3.4, we got $(f(xy), f(x)f(y)) \in \beta$.

Conversely, by the fact $f(1) = 1$ we have the sequence of equivalent equations: the equation $(xy \otimes xy) = 1$ by assertion (2) of Theorem 3.6 ([24]) is equivalent to $f(x \otimes (y \otimes xy)) = 1$ and, since f is a homomorphism, $f(x) \otimes f(y \otimes xy) = 1$, i.e. it is equivalent to the equation $f(x) \otimes (f(y) \otimes f(xy)) = 1$. Thus, again by using assertion (2) of Theorem 3.6 ([24]), we have the equivalent equation $(f(x)f(y)) \otimes f(xy) = 1$. The last equation means $(f(x)f(y), f(xy)) \in \beta$. Therefore, we have proved that $(f(x)f(y), f(xy)) \in \beta \cup \beta^{-1} \neq \neq$. Since the apartness is tight, we finally have

$$f(x)f(y) = f(xy).$$

(5) Let u be an arbitrary element of G . Then $f(u) \in \{1\}^C$, i.e. holds $f(u) \neq 1$. Thus, $u \neq 1$. So, we have $1 \bowtie G$. Further on, let x, y elements of S such that $y \in G$. Then, from $f(y) \neq 1$ follows $f(y) \neq f(x \otimes y) \vee f(x \otimes y) \neq 1$. Out of the first part, i.e. from $1 \otimes f(y) \neq f(x) \otimes f(y)$ by the comment before this theorem we conclude $1 \neq f(x)$. So, we have $x \in G$ or $x \otimes y \in G$. Therefore, the set G is an ordered anti-filter in S . The last assertion is clear because f is a strongly extensional mapping.

(6) Suppose that f is an embedding function from S in T . Then the implication $x \neq 1 \implies f(x) \neq 1$ is true. So, holds $\{1\}^C \subseteq G \subseteq \{1\}^C$.

Let the equation $\{1\}^C = f^{-1}(\{1\}^C)$ be true and let $x \neq y$ hold for elements $x, y \in S$. Then we have $(x, y) \in \alpha \vee (y, x) \in \alpha$, and thus $x \otimes y \neq 1 \vee y \otimes x \neq 1$. The last means $x \otimes y \in \{1\}^C = f^{-1}(\{1\}^C)$ or $y \otimes x \in \{1\}^C = f^{-1}(\{1\}^C)$. Hence, we have $f(x \otimes y) \neq 1 \vee f(y \otimes x) \neq 1$, i.e. we have $f(x) \otimes f(y) \neq 1$ or $f(y) \otimes f(x) \neq 1$. So, we have $(f(x), f(y)) \in \beta \cup \beta^{-1} \neq \neq$. This proves that f is an embedding. \square

Moreover, we can prove more then in assertion (5) of above theorem.

Theorem 3.2. *Let f be as in Theorem 3.1. If G is an ordered anti-filter in T , then the set $f^{-1}(G)$ is an ordered anti-filter in S .*

Proof. Let u be an arbitrary element of $f^{-1}(G)$. Then, $f(u) \neq 1$, and thus $u \neq 1$. So, $1 \bowtie f^{-1}(G)$. Let x and y be element of S such that $y \in f^{-1}(G)$. Since, the mapping f is surjective, from $f(y) \in G$ follows $f(x) \otimes f(y) \in G$ or $f(x) \in G$. Hence, we have $f(x \otimes y) \in G \vee f(x) \in G$, and finally $x \otimes y \in f^{-1}(G)$ or $x \in f^{-1}(G)$. Hence, the set $f^{-1}(G)$ is an ordered anti-filter in S . \square

Let f be as in the Theorem 3.1. Then $f^{-1}(\beta) = \{(x, y) \in S \times S : (f(x), f(y)) \in \beta\}$ is a quasi-antiorder relation on S compatible with the semigroup operation in S . Moreover, the relation $q = f^{-1}(\beta) \cup (f^{-1}(\beta))^{-1}$ is a coequality relation on S compatible with the semigroup operation on S . Further on, the semigroup $((S/q, =_1, \neq_1), \cdot)$ is an anti-ordered semigroup ordered under anti-order θ , defined by $(xq, yq) \in \theta \iff (x, y) \in f^{-1}(\beta)$. It is not so hard to check that the semigroup S/q is negatively anti-ordered under θ .

We now define another operation \otimes_1 on semigroup S/q in the following way:

$$xq \otimes_1 yq \equiv (x \otimes y)q \text{ (for any } x, y \in S\text{.)}$$

In the following lemma we show that the operation ' \otimes_1 ' is well-defined.

Lemma 3.1. *Let f be as in the Theorem 3.1. Then $'\otimes_1'$ is extensional and strongly extensional function from $S/q \times S/q$ into S/q such that the anti-order θ is compatible anti-order on S/q . Hence, the semigroup $((S/q, =_1, \neq_1), \cdot, \theta, \otimes_1)$ is an implicative semigroup.*

Proof. (1) Let $xq, yq, x'q, y'q$ be arbitrary elements of S/q such that $xq =_1 x'q$ and $yq =_1 y'q$, i.e. holds $(x, x') \bowtie q$ and $(y, y') \bowtie q$. Assume that (u, v) be an arbitrary element of q . Then, we have

$$(u, x \otimes y) \in q \vee (x \otimes y, x' \otimes y) \in q \vee (x' \otimes y, x' \otimes y') \in q \vee (x' \otimes y', v) \in q.$$

Since from the second and third part we conclude $(x, x') \in q$ and $(y, y') \in q$ which is impossible, finally we have $(x \otimes y, x' \otimes y') \neq (u, v) \in q$. Last means $(x \otimes y)q =_1 (x' \otimes y')q$. Therefore, the equality $xq \otimes_1 yq =_1 x'q \otimes_1 y'q$ is true.

(2) Suppose that $xq \otimes_1 yq \neq_1 x'q \otimes_1 y'q$ holds, i.e. assume that $(x \otimes y)q \neq_1 (x' \otimes y)q$ holds. This means $(x \otimes y, x' \otimes y) \in q$. Thus, we have $(x, x') \in q$. So, we have $xq \neq_1 x'q$. The implication $xq \otimes_1 yq \neq_1 xq \otimes_1 y'q \implies yq \neq_1 y'q$ we prove analogously. So, the function $'\otimes_1'$ is a strongly extensional function.

Therefore, the operation $'\otimes_1'$ is well-defined.

(3) Let xq, yq, zq be arbitrary elements of S/q such that $(xq \cdot yq, yq \cdot zq) \in \theta$. Then, $(xz, yz) \in f^{-1}(\beta)$, and hence $(x, y) \in f^{-1}(\beta)$ since $f^{-1}(\beta)$ is compatible anti-order relation with the semigroup operation in S . So, we have $(xq, yq) \in \theta$. Similarly, we can prove that $(zq \cdot xq, zq \cdot yq) \in \theta$ implies $(xq, yq) \in \theta$ in a similar way. Hence, we have proved that the anti-order θ is compatible with the semigroup operation in S/q . □

The following theorem is an adapted version of a standard theorem on homomorphism (see for example [20] and [21]) for our needs.

Theorem 3.3. *Let $f : S \longrightarrow T$ be a strongly extensional homomorphism from an implicative semigroup S onto an implicative semigroup T . Then there exists a strongly extensional homomorphism g from the implicative semigroup S/q onto the semigroup T such that $f = g \circ \pi$, where $\pi : S \longrightarrow S/q$ is the natural homomorphism.*

Proof. The proof of this theorem we can get by analogous procedures of the proof of Theorem 4 in paper [21]. □

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Daniel A. Romano
 Kordunaška 6
 78000 Banja Luka
 Bosnia and Herzegovina
 bato49@hotmail.com