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SUMMATION PROCESS OF CONVOLUTION OPERATORS FOR MULTIVARIABLES

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Abstract. In this paper, we study Korovkin type approximation results for a sequence of positive linear convolution operators defined on $C([a,b]\times[c,d])$, the space of all continuous real valued functions on $[a,b]\times[c,d]$, with the use of $\mathcal{A}$-summation process which includes both convergence and almost convergence. We also study rate of convergence of these operators.

1. Introduction

The classical theorem of Korovkin [11] on approximation of continuous functions on a compact interval gives conditions in order to make a decision whether a sequence of positive linear operators converges to the identity operator. Most of the classical approximation operators tend to converge to value of function being approximated. However, at points of discontinuity, they often converge to the average of the left and right limits of the functions. There are, however, some exceptions that do not converge at points of simple discontinuity [5]. The main purpose of using summability theory has always been to make a nonconvergent sequence converge. The purpose of this paper is to study a Korovkin type approximation of a function f by means of a sequence $\{L_j(f;x,y)\}$ of positive linear convolution operators over $C([a,b]\times[c,d])$ with the use of a matrix summability method which includes both convergence and almost convergence. We also study the rate of convergence of these operators.

Approximation theory has important applications in various areas of functional analysis, in numerical solutions of differential and integral equations [2], [6], [11]. Some unification on Korovkin type results through the use of a summability method

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may be found in [1], [3], [8], [10], [13], [14] and [16] . Convolution type approximation theory may be found in [7], [18], [19].

We first recall some notation and basic definitions used in this paper.

As usual $C([a,b]\times[c,d])$, $B([a,b]\times[c,d])$ denote the space of all continuous real valued functions and all bounded real valued functions on $[a,b]\times[c,d]$, respectively. Note that $C([a,b]\times[c,d])$ is a Banach space with the norm $\|.\|$ defined by

$$\|f\| := \sup_{(x,y)\in[a,b]\times[c,d]} |f(x,y)|\,,\ \ f\in C([a,b]\times[c,d]).$$

Let $\mathcal{A} := \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries. A sequence $\{L_j\}$ of positive linear operators of $C([a,b]\times[c,d])$ into $B([a,b]\times[c,d])$ is called an $\mathcal{A}-$summation process on $C([a,b]\times[c,d])$ if $\{L_j(f)\}$ is $\mathcal{A}-$summable to f for every $f\in C([a,b]\times[c,d])$, i.e.,

$$\lim_k \left\| \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f) - f \right\| = 0,$$

uniformly in n, where it is assumed that the series converges for each k,n and f. Recall that a sequence of real numbers $\{x_j\}$ is said to be $\mathcal{A}-$summable to l if $\lim\limits_k \sum\limits_{j=1}^{\infty} a_{kj}^{(n)} x_j = l$, uniformly in n ([4], [15]).

Recall that $\mathcal{A} := \left\{A^{(n)}\right\}$ is regular [4] if and only if

$i)$ for each $j=1,2,...,$ $\lim\limits_k a_{kj}^{(n)} = 0$, uniformly in n.

$ii)$ $\lim\limits_k \sum\limits_j a_{kj}^{(n)} = 1$, uniformly in n.

$iii)$ for each $n,k=1,2,...,$ $\sum\limits_j \left|a_{kj}^{(n)}\right| < \infty$, and there exist an integer N and a constant M such that $\sum\limits_j \left|a_{kj}^{(n)}\right| < M$ for $k\geq N$ and all $n=1,2,....$

If $A^{(n)} = A$, $n=1,2,3,...$ for some matrix A, then $\mathcal{A}$-summability is the ordinary matrix summability by A.

Throughout this paper we assume that $\mathcal{A}$ is regular.

Let $J := [a,b]\times[c,d]$ and L be a linear operator from $C(J)$ into $C(J)$. Then we say that L is a positive linear operator provided that $f\geq 0$ implies $Lf\geq 0$. Also we denote the value of $L(f)$ at a point $(x,y)\in J$ by $L(f;x,y)$.

We now consider the convolution operators

$$L_j(f;x,y) = \int_c^d \int_a^b f(u,v) K_j(u-x, v-y)\,du\,dv$$

where $x \in [a,b]$, $y \in [c,d]$, $f \in C(J)$. Throughout the paper we assume that K_j is a continuous function on $[a-b,b-a] \times [c-d,d-c]$ and also that $K_j(t,z) \geq 0$ for all $n \in \mathbb{N}$ and for every $(t,z) \in [a-b,b-a] \times [c-d,d-c]$. Note that if $u,x \in [a,b]$ and $v,y \in [c,d]$ then $t := u-x \in [a-b,b-a]$ and $z := v-y \in [c-d,d-c]$. In this case our convolution operators L_j are positive and linear.

Let $\{L_j\}$ be a sequence of positive linear operators from $C(J)$ into $C(J)$ such that for each $k,n \in \mathbb{N}$

$$\sum_{j=1}^{\infty} a_{kj}^{(n)} \left\| L_j(1) \right\| < \infty. \tag{1.1}$$

Furthermore, for each $k,n \in \mathbb{N}$ and $f \in C(J)$, let

$$B_k^{(n)}(f;x,y) = \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(f;x,y)$$

which is well defined by (1.1), and belongs to $B(J)$.

The Korovkin theory for multivariables has been studied in [9].

2. A Korovkin-Type Approximation Theorem

In this section, using $\mathcal{A}-$summation process, we give Korovkin type approximation of a function f by means of a sequence $\{L_j(f;x,y)\}$ of positive linear convolution operators over $C([a,b] \times [c,d])$. The next result may be obtained by a slight modification of Theorem 2.1 of [7].

Theorem 1. *Let $\mathcal{A} = \left\{A^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries. Assume that $\{L_j\}$ is a sequence of positive linear operators from $C(J)$ into $C(J)$ for which (1.1) holds. If*

$$\lim_k \left\| B_k^{(n)}(f_0) - f_0 \right\| = 0 \text{ with } f_0(y) = 1, \text{ uniformly in } n$$

and

$$\lim_k \left\| B_k^{(n)}(\Psi) \right\| = 0 \text{ with } \Psi(u,v) = (u-x)^2 + (v-y)^2, \text{ uniformly in } n$$

then for all $f \in C(J)$, we have

$$\lim_k \left\| B_k^{(n)}(f) - f \right\| = 0, \text{ uniformly in } n.$$

Let

$$\|f\|_\delta := \sup_{\substack{a+\delta \leq x \leq b-\delta \\ c+\delta \leq y \leq d-\delta}} |f(x,y)|, \; f \in C(J)$$

where δ is a positive real number such that $\delta < \min\left\{\frac{b-a}{2}, \frac{d-c}{2}\right\}$, and let $B_\eta := [a-b, b-a] \times [c-d, d-c] \setminus [-\eta, \eta] \times [-\eta, \eta]$ for any $\eta > 0$ satisfying $\eta < \min\{b-a, d-c\}$.

In order to give our results we need the following

Lemma 1. *Let $\mathcal{A} = \left\{A^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries and δ be a fixed positive real number such that $\delta < \min\left\{\frac{b-a}{2}, \frac{d-c}{2}\right\}$. Assume that $\{L_j\}$ is a sequence of positive linear operators from $C(J)$ into $C(J)$ for which (1.1) holds. If*

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \left\{ \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_j(u,v)\,du\,dv \right\} = 1, \text{ uniformly in } n$$

and

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \left\{ \sup_{(u,v)\in B_\eta} K_j(u,v) \right\} = 0, \text{ uniformly in } n, \text{ for any } \eta > 0$$

then we have

$$\lim_k \left\| B_k^{(n)}(f_0) - f_0 \right\|_\delta = 0, \text{ uniformly in } n.$$

Proof. Fix $0 < \delta < \min\left\{\frac{b-a}{2}, \frac{d-c}{2}\right\}$ and let $x \in [a+\delta, b-\delta]$, $y \in [c+\delta, d-\delta]$. Then it is easy to see that

$$\begin{aligned} -(b-a) &\le a - x \le -\delta \\ -(d-c) &\le c - y \le -\delta \\ \delta &\le b - x \le b - a \\ \delta &\le d - y \le d - c. \end{aligned}$$

We have, for all $n \in \mathbb{N}$, that

$$B_k^{(n)}(f_0; x, y) = \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_c^d \int_a^b K_j(u-x, v-y)\,du\,dv = \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{c-y}^{d-y} \int_{a-x}^{b-x} K_j(u,v)\,du\,dv.$$

Hence we get

$$\sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_j(u,v)\,du\,dv \le B_k^{(n)}(f_0; x, y) \le \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{-(d-c)}^{d-c} \int_{-(b-a)}^{b-a} K_j(u,v)\,du\,dv.$$

These inequalities imply that

$$\left\| B_k^{(n)}(f_0) - f_0 \right\|_\delta \le z_k^{(n)}$$

where

$$z_k^{(n)} = \max\left\{ \left| \left\{ \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_j(u,v)dudv \right\} - 1 \right|, \left| \left\{ \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{-(d-c)}^{d-c}\int_{-(b-a)}^{b-a} K_j(u,v)dudv \right\} - 1 \right| \right\}.$$

Notice that by the hypothesis, it is clear that

$$\lim_k z_k^{(n)} = 0, \text{ uniformly in } n.$$

Taking limit as $k \to \infty$ the result follows. □

Lemma 2. *Let $\mathcal{A} = \left\{A^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries and δ be a fixed positive real number such that $\delta < \min\left\{\frac{b-a}{2}, \frac{d-c}{2}\right\}$. Assume that $\{L_j\}$ is a sequence of positive linear operators from $C(J)$ into $C(J)$ for which (1.1) holds. If*

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \left\{ \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} K_j(u,v)dudv \right\} = 1, \text{ uniformly in } n$$

and

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \left\{ \sup_{(u,v)\in B_\eta} K_j(u,v) \right\} = 0, \text{ uniformly in } n, \text{ for any } \eta > 0$$

then we have

$$\lim_k \left\| B_k^{(n)}(\Psi) \right\|_\delta = 0, \text{ uniformly in } n$$

where $\Psi(u,v) = (u-x)^2 + (v-y)^2$.

Proof. Fix $0 < \delta < \min\left\{\frac{b-a}{2}, \frac{d-c}{2}\right\}$ and let $x \in [a+\delta, b-\delta]$, $y \in [c+\delta, d-\delta]$. Note that, for $x \in [a+\delta, b-\delta]$, since $\Psi(u,v) = (u-x)^2 + (v-y)^2$, it is obvious that $\Psi \in C(J)$.

$$\begin{aligned} B_k^{(n)}(\Psi; x, y) &= \sum_{j=1}^{\infty} a_{kj}^{(n)} L_j(\Psi; x, y) \\ &= \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_c^d \int_a^b \left[(u-x)^2 + (v-y)^2\right] K_j(u-x, v-y)dudv \\ &= \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{c-y}^{d-y}\int_{a-x}^{b-x} \left[u^2+v^2\right] K_j(u,v)dudv \\ &\le \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{-(d-c)}^{d-c}\int_{-(b-a)}^{b-a} \left[u^2+v^2\right] K_j(u,v)dudv. \end{aligned}$$

By the continuity of Ψ at $(0,0)$, for sufficiently small $\varepsilon > 0$, $(0 < \sqrt{\varepsilon} < \delta)$, $\Psi(u,v) < 2\varepsilon$ whenever $|u| < \sqrt{\varepsilon}$, $|v| < \sqrt{\varepsilon}$. Then we may write that

$$\begin{aligned}
B_k^{(n)}(\Psi; x, y) &\leq 2\varepsilon \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} K_j(u,v)\,dudv + \sum_{j=1}^{\infty} a_{kj}^{(n)} \iint_{B_{\sqrt{\varepsilon}}} \left[u^2 + v^2\right] K_j(u,v)\,dudv \\
&\leq 2\varepsilon \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} K_j(u,v)\,dudv \\
&\quad + \sum_{j=1}^{\infty} a_{kj}^{(n)} \left\{ \sup_{(u,v)\in B_{\sqrt{\varepsilon}}} K_j(u,v) \right\} \int_{c-d}^{d-c} \int_{a-b}^{b-a} \left[u^2 + v^2\right] dudv \\
&= 2\varepsilon \sum_{j=1}^{\infty} a_{kj}^{(n)} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} K_j(u,v)\,dudv + M \sum_{j=1}^{\infty} a_{kj}^{(n)} \left\{ \sup_{(u,v)\in B_{\sqrt{\varepsilon}}} K_j(u,v) \right\},
\end{aligned}$$

where $M = \int_{c-d}^{d-c} \int_{a-b}^{b-a} \left[u^2 + v^2\right] dudv$. Now taking limit as $k \to \infty$ (uniformly in n) and also using the hypotheses, the proof is completed. □

Now we are ready to give our main result.

Theorem 2. *Let $\mathcal{A} = \left\{A^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries and δ be a fixed positive real number such that $\delta < \min\left\{\frac{b-a}{2}, \frac{d-c}{2}\right\}$. Assume that $\{L_j\}$ is a sequence of positive linear operators from $C(J)$ into $C(J)$ for which (1.1) holds. If*

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \left\{ \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} K_j(u,v)\,dudv \right\} = 1, \text{ uniformly in } n$$

and

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \left\{ \sup_{(u,v)\in B_\eta} K_j(u,v) \right\} = 0, \text{ uniformly in } n, \text{ for any } \eta > 0$$

then we have

$$\lim_k \left\| B_k^{(n)}(f) - f \right\|_\delta = 0, \text{ uniformly in } n.$$

Example 1. *We now exhibit a sequence of positive convolution operators satisfying Theorem 2. In order to see this let $\mathcal{A} = \left\{A^{(n)}\right\} = \left\{a_{kj}^{(n)}\right\}$ be a sequence of infinite matrices defined by $a_{kj}^{(n)} = 1/k$ if $n \leq j < n+k$, and $a_{kj}^{(n)} = 0$ otherwise. In this case $\mathcal{A}-$summability method reduces to almost convergence introduced by Lorentz*

[12].

Now let the operators L_j from $C(J)$ into $C(J)$ be defined by

$$L_j(f;x,y)=\frac{j^2}{\pi}\int_c^d\int_a^b f(u,v)e^{-j^2(u-x)^2}e^{-j^2(v-y)^2}dudv.$$

If we choose $(d_j)=\left\{(-1)^j\right\}$, which is almost convergent to zero, but not convergent, and define

$$T_j(f;x,y)=(1+d_j)L_j(f;x,y).$$

Then the operators T_j given above have the form of the convolution operators where

$$K_j(u,v)=\frac{(1+d_j)j^2}{\pi}e^{-j^2u^2}e^{-j^2v^2}.$$

For every $\delta>0$ such that $\delta<\min\left\{\frac{b-a}{2},\frac{d-c}{2}\right\}$, we have

$$\sum_j a_{kj}^{(n)}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}K_j(u,v)dudv=$$

$$\sum_j a_{kj}^{(n)}\frac{(1+d_j)j^2}{\pi}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-j^2u^2}e^{-j^2v^2}dudv-\iint_{(u,v)\in U_\delta}e^{-j^2u^2}e^{-j^2v^2}dudv\right\}\quad(2.1)$$

where $U_\delta:=\{(u,v):|u|\geq\delta \text{ or } |v|\geq\delta\}$. Since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-j^2u^2}e^{-j^2v^2}dudv=\frac{1}{j^2}\pi<\infty,$$

it is clear that

$$\lim_k\iint_{(u,v)\in U_\delta}e^{-k^2u^2}e^{-k^2v^2}dudv=0.$$

Also, since $\lim_k\sum_{j=1}^{\infty}a_{kj}^{(n)}d_j=0$, uniformly in n, we immediately get

$$\lim_k\sum_{j=1}^{\infty}a_{kj}^{(n)}\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}K_j(u,v)dudv=1,\text{ uniformly in } n.$$

One can easily see that for any $\eta>0$,

$$\sup_{(u,v)\in B_\eta}K_j(u,v)\leq\frac{(1+d_j)j^2}{\pi}\frac{1}{e^{j^2\eta^2}}.$$

Since

$$\lim_j \frac{j^2}{e^{j^2\eta^2}} = 0 \text{ and } \lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)}(1+d_j) = 1, \text{ uniformly in } n,$$

we conclude that

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \left\{ \sup_{(u,v)\in B_\eta} K_j(u,v) \right\} = 0, \text{ uniformly in } n.$$

So the conditions in Theorem 2 are satisfied.

3. Rate of Convergence

In this section, using the modulus of continuity, we study the rate of convergence in the last theorem.

For the functions of two variables, it is known that there are different types of modulus of continuity [17].

We now turn to introducing some notation and basic definitions to obtain the rate of $\mathcal{A}$-summation process of the operators L_n. Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a continuous function and β be a positive number. Full continuity modulus of the function $f(x,y)$ is defined by

$$w(f;\beta) := \max_{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \leq \beta} \left| f(x_1,y_1) - f(x_2,y_2) \right|.$$

It is also known that

$$\lim_{\beta\to 0} w(f;\beta) = 0 \text{ and for any } \lambda > 0,\ w(f;\lambda\beta) \leq ([|\lambda|]+1)w(f;\beta)$$

where $[|c|]$ is defined to be the greatest integer less than or equal to c [17].

Theorem 3. *Let $\mathcal{A} = \left\{A^{(n)}\right\}$ be a sequence of infinite matrices with nonnegative real entries and δ be a fixed positive real number such that $\delta < \min\left\{\frac{b-a}{2}, \frac{d-c}{2}\right\}$. Assume that $\{L_j\}$ is a sequence of positive linear operators from $C(J)$ into $C(J)$ for which (1.1) holds then for all $f \in C(J)$, we have the following inequality*

$$\left\| B_k^{(n)} f - f \right\|_\delta \leq M \left\{ w(f;\alpha_k^{(n)}) + w(f;\alpha_k^{(n)}) \left\| B_k^{(n)} f_0 - f_0 \right\|_\delta + \left\| B_k^{(n)} f_0 - f_0 \right\|_\delta \right\}$$

where $\alpha := \alpha_k^{(n)} = \sqrt{\left\| B_k^{(n)}((u-x)^2+(v-y)^2;x,y) \right\|_\delta}$.

Proof. Let $0 < \delta < \min\left\{\frac{b-a}{2}, \frac{d-c}{2}\right\}$, $f \in C(J)$ and $x \in [a+\delta, b-\delta]$, $y \in [c+\delta, d-\delta]$. By positivity and linearity of the operators L_n, we get for any $\alpha > 0$, that

$$\begin{aligned}
&\left|B_k^{(n)}(f;x,y) - f(x,y)\right| = \left|B_k^{(n)}(f(u,v) - f(x,y);x,y) + f(x,y)(B_k^{(n)}f_0 - f_0)\right| \\
&\leq B_k^{(n)}(\left|f(u,v) - f(x,y)\right|;x,y) + |f(x,y)|\left|B_k^{(n)}f_0 - f_0\right| \\
&\leq B_k^{(n)}(w(f;\alpha\frac{\sqrt{(u-x)^2+(v-y)^2}}{\alpha};x,y) + |f(x,y)|\left|B_k^{(n)}f_0 - f_0\right| \\
&\leq w(f;\alpha)B_k^{(n)}(1 + \left[\left|\frac{\sqrt{(u-x)^2+(v-y)^2}}{\alpha}\right|\right];x,y) + |f(x,y)|\left|B_k^{(n)}f_0 - f_0\right| \\
&\leq w(f;\alpha)B_k^{(n)}(1 + \frac{(u-x)^2+(v-y)^2}{\alpha^2};x,y) + |f(x,y)|\left|B_k^{(n)}f_0 - f_0\right| \\
&\leq w(f;\alpha)\left\{B_k^{(n)}f_0 + \frac{1}{\alpha^2}B_k^{(n)}((u-x)^2+(v-y)^2;x,y)\right\} + |f(x,y)|\left|B_k^{(n)}f_0 - f_0\right|.
\end{aligned}$$

This yields that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\left\|B_k^{(n)}f - f\right\|_\delta &\leq w(f;\alpha)\left\{\left\|B_k^{(n)}f_0\right\|_\delta + \frac{1}{\alpha^2}\left\|B_k^{(n)}((u-x)^2+(v-y)^2;x,y)\right\|_\delta\right\} \\
&\quad + M_1\left\|B_k^{(n)}f_0 - f_0\right\|_\delta \qquad (3.1)
\end{aligned}$$

where $M_1 := \|f\|_\delta$. Now letting $\alpha := \alpha_k^{(n)} = \sqrt{\left\|B_k^{(n)}((u-x)^2+(v-y)^2;x,y)\right\|_\delta}$, we have

$$\begin{aligned}
\left\|B_k^{(n)}f - f\right\|_\delta &\leq w(f;\alpha_k^{(n)})\left\{\left\|B_k^{(n)}f_0\right\|_\delta + 1\right\} + M_1\left\|B_k^{(n)}f_0 - f_0\right\|_\delta \\
&\leq 2w(f;\alpha_k^{(n)}) + w(f;\alpha_k^{(n)})\left\|B_k^{(n)}f_0 - f_0\right\|_\delta + M_1\left\|B_k^{(n)}f_0 - f_0\right\|_\delta.
\end{aligned}$$

Let $M := \max\{2, M_1\}$. Then we can write, for all $n \in \mathbb{N}$, that

$$\left\|B_k^{(n)}f - f\right\|_\delta \leq M\left\{w(f;\alpha_k^{(n)}) + w(f;\alpha_k^{(n)})\left\|B_k^{(n)}f_0 - f_0\right\|_\delta + \left\|B_k^{(n)}f_0 - f_0\right\|_\delta\right\}$$

from which the result follows. □

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