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APPROXIMATION BY MEANS OF FOURIER TRIGONOMETRIC SERIES IN WEIGHTED LEBESGUE SPACES

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Abstract. In this work the approximation of the functions by trigonometric polynomials $N_n^\lambda(f;x)$ of degree n in weighted Lebesgue spaces with Muckenhoupt weights are studied.

1. Introduction and The Main Results

Let $\mathbb{T}$ denote the interval $[-\pi,\pi]$. We denote by $L^p(\mathbb{T})$, $1\leq p<\infty$, the Lebesgue space of all measurable functions f on $\mathbb{T}$, that is, the space of all such functions for which

$$\|f\|_{L^p(\mathbb{T})}=\left(\int\limits_{\mathbb{T}}|f(x)|^p\,dx\right)^{1/p}<\infty.$$

A function ω is called a weight on $\mathbb{T}$ if $\omega:\mathbb{T}\longrightarrow[0,\infty]$ is measurable and $\omega^{-1}(\{0,\infty\})$ has measure zero (with respect to Lebesgue measure). With any given ω we associate the $\omega-$*weighted Lebesgue space* $L_\omega^p:=L_\omega^p(\mathbb{T})$ consisting of all measurable functions f on $\mathbb{T}$ such that

$$\|f\|_{L_\omega^p(\mathbb{T})}=\left(\int\limits_{\mathbb{T}}|f(x)|^p\,\omega(x)dx\right)^{1/p}.$$

Let $1<p<\infty$ and let $A_p(\mathbb{T})$ be the collection of all weights on $\mathbb{T}$ satisfying the condition

$$\sup_I\left(\frac{1}{|I|}\int_I\omega(x)dx\right)\left(\frac{1}{|I|}\int_I[\omega(x)]^{-1/p-1}dx\right)^{p-1}<\infty$$

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where the supremum is taken over all intervals I with length $|I| \leq 2\pi$. The condition is called the *Muckenhoupt* -A_p condition [15] and the weight functions which belong to $A_p(\mathbb{T})$, $(1 < p < \infty)$, are called the *Muckenhoupt weights.*

Let $1 < p < \infty$ and let $\omega \in A_p(\mathbb{T})$. For $f \in L^p_\omega$ we define the modulus of continuity $\Omega(f, \cdot,)_{L^p_\omega}$ by

$$\Omega(f, \cdot)_{L^p_\omega} = \sup_{|h| \leq \delta} \|\Delta_h(f)\|_{L^p_\omega} \quad \delta > 0$$

where

$$\Delta_h(f)(x) = \frac{1}{h} \int_0^h |f(x+t) - f(x)|\, dt.$$

Note that the modulus of continuity $\Omega(f, \cdot,)_{L^p_\omega}$, defined by N. X. Ky [12]. The modulus of continuity $\Omega(f, \cdot)_{L^p_\omega}$ is a nondecreasing, nonnegative, continuous function and

$$\lim_{\delta \to 0} \Omega(f, \cdot)_{L^p_\omega} = 0, \quad \Omega(f+g, \cdot)_{L^p_\omega} \leq \Omega(f, \cdot)_{L^p_\omega} + \Omega(g, \cdot)_{L^p_\omega},$$

for $f, g \in L^p_\omega$.

Let $0 < \alpha \leq 1$. The set of functions $f \in L^p_\omega$ such that

$$\Omega(f, \delta)_{L^p_\omega} = O(\delta^\alpha), \quad \delta > 0$$

is called the *Lipschitz class* $Lip(\alpha, p, \omega)$. Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k\left(f\right) \cos kx + b_k\left(f\right) \sin kx\right) \tag{1}$$

be the Fourier series of the function $f \in L^1(\mathbb{T})$, where $\alpha_k(f)$ are $\beta_k(f)$ the Fourier coefficients of the function f. The *n-th partial sum* of series (1) is defined, as

$$\begin{aligned} S_n\left(f; x\right) &= \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k\left(f\right) \cos kx + b_k\left(f\right) \sin kx\right), \\ &= \sum_{k=0}^{n} Q_k(f; x). \end{aligned}$$

Let $\{p_n\}_0^\infty$ be a sequence of positive real numbers. The sequence $\{p_n\}_0^\infty$ is called *almost monotone decreasing (increasing)*, denoted by $\{p_n\}_0^\infty \in AMDS$ $(\{p_n\}_0^\infty \in AMIS)$, if there exist a constant c, depending only on the sequence $\{p_n\}_0^\infty$ such that for all $n \geq m$ the following inequality holds:

$$p_n \leq c p_m, \qquad (p_m \leq c p_n,)$$

In proof of the main result we will use the notations

$$\Delta\beta_n := \beta_n - \beta_{n+1}, \quad \Delta_m \beta(n, m) := \beta(n, m) - \beta(n, m+1)$$

As in [20] we suppose that $\mathbb{F}$ is an infinite subset of $\mathbb{N}$ and consider $\mathbb{F}$ as the range of strictly increasing sequence of positive integers, say $\mathbb{F} = \{\lambda(n)\}_1^\infty$ .Following [1], [22] the Cesáro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \; n = 1, 2, ...,$$

where $\{x_k\}$ is a sequence of a real or complex numbers. Therefore, the $C_\lambda-$ method yields a subsequence of the Cesáro method C_1, and hence it is regular for any λ. C_λ is obtained by deleting a set of rows from Cesáro matrix. We suppose that $\{p_n\}_0^\infty$ is a sequence of positive real numbers. We define the mean of the series (1), as

$$N_n^\lambda(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{n} p_{\lambda(n)-m} s_m(f;x)$$

where $P_n := \sum_{m=0}^{n} p_m \neq 0 \;\; (n \geq 0), \;\; p_{-1} = P_{-1} = 0$. Note that in the case $p_n = 1$, $n \geq 0$, $N(f;x)$ is equal to the mean

$$\sigma_n^\lambda(f;x) = \frac{1}{\lambda(n)+1} \sum_{m=0}^{\lambda(n)} S_m(f;x).$$

In the present paper we study the approximation of $f \in L_\omega^p$ by trigonometric polynomials $N_n^\lambda(f;x)$. We give the weighted versions of the results obtained in [20] in the case $1 < p < \infty$. Similar problems about approximations of the functions by trigonometric polynomials in the different spaces have been investigated by several authors (see, for example, [2-11], [13-15], [17-21] and [23-25]).

Note that, in the proof of the main results we use the method as in the proof of [20].

Our main result is the following:

Theorem 1. *Let $1 < p < \infty$, $\omega \in A_p(\mathbb{T})$ and let $\{p_n\}_0^\infty$ be a sequence of positive real numbers. If $f \in Lip(\alpha, p, \omega)$ and if one of the conditions*

$$\text{(i)} \quad p > 1, \; 0 < \alpha < 1, \; \textit{and} \; \{p_n\}_0^\infty \in AMDS$$

$$\text{(ii)} \quad p > 1, \; 0 < \alpha < 1, \; \textit{and} \; \{p_n\}_0^\infty \in AMIS,$$

and

$$(\lambda(n)+1)p_{\lambda(n)} = O(P_{\lambda(n)}) \tag{2}$$

holds,

$$\text{iii)}\quad p>1,\alpha=1,\ \sum_{k=!}^{\lambda(n)-1} k\left|\Delta p_k\right| = O(P_{\lambda(n)}),$$

$$\text{(iv)}\quad p>1,\alpha=1,\ \sum_{k=!}^{\lambda(n)-1} \left|\Delta p_k\right| = O(P_{\lambda(n)}/\lambda(n)),\ \textit{and (2) holds,}$$

then the estimate

$$\left\|f-N_n^{\lambda}(f)\right\|_{L^p_\omega(\mathbb{T})} = O\left((\lambda(n)))^{-\alpha}\right).$$

holds.

In the proof of the main result we need the following Lemmas:

Lemma 1. (see [5]). *Let* $1<p<\infty$, $\omega\in A_p(\mathbb{T})$ *and* $0<\alpha<1$. *Then for* $f\in Lip(\alpha,p,\omega)$ *and* $n=1,\ 2,...$ *the estimate*

$$\left\|f-S_n(f)\right\|_{L^p_\omega} = O(n^{-\alpha})$$

holds.

Lemma 2. (see [5]). *Let* $1<p<\infty$, *and* $\omega\in A_p(\mathbb{T})$. *Then for* $f\in Lip(1,p,\omega)$ *and* $n=1,\ 2,...$ *the estimate*

$$\left\|S_n(f)-\sigma_n(f)\right\|_{L^p_\omega} = O(n^{-1})$$

holds.

Lemma 3. (see [20]). *If* $\{p_n\}_0^\infty\in AMDS$ *or* $\{p_n\}_0^\infty\in AMIS$ *and (2) holds, then*

$$\sum_{m=1}^{\lambda(n)} m^{-\alpha}p_{\lambda(n)-m} = O\left((\lambda(n))^{-\alpha}P_{\lambda(n)}\right)$$

for $0<\alpha<1$.

2. Proofs of the main results

Proof of Theorem 1. We prove the cases (i) and (ii) together. It is clear that

$$N_n^{\lambda}(f;x)-f(x) = \frac{1}{P_{\lambda(n)}}\sum_{m=o}^{\lambda(n)} p_{\lambda(n)-m}\left\{s_m(f;x)-f(x)\right\}. \tag{3}$$

Then by Lemma 1 and Lemma 3 and (3) and condition (2) we find

$$\begin{aligned}\left\|N_n^\lambda(f)-f\right\|_{L^p_\omega} &\le \frac{1}{P_{\lambda(n)}}\sum_{m=o}^{\lambda(n)} p_{\lambda(n)-m}\left\|f-s_m(f)\right\|_{L^p_\omega}\\ &= \frac{1}{P_{\lambda(n)}}\sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m}\left\|f-s_m(f)\right\|_{L^p_\omega}+\left\|f-s_0(f)\right\|_{L^p_\omega}\\ &= \frac{1}{P_{\lambda(n)}}\sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m}O(m^{-\alpha})+O(\frac{p_{\lambda(n)}}{P_{\lambda(n)}})\\ &= O((\lambda(n))^{-\alpha}).\end{aligned}$$

Case (iv): We suppose that $p>1$ and $\alpha=1$. Using Abel's transformation, we have

$$N_n^\lambda(f;x)=\frac{1}{P_{\lambda(n)}}\sum_{m=o}^{\lambda(n)} p_{\lambda(n)-m}\left\{s_m(f;x)-f(x)\right\}Q_m(f;x).$$

Thus

$$s_n^\lambda(f;x)-N_n^\lambda(f;x)=\frac{1}{P_{\lambda(n)}}\sum_{m=1}^{\lambda(n)}(P_{\lambda(n)}-P_{\lambda(n)-m})\left\{s_m(f;x)-f(x)\right\}Q_m(f;x).$$

By Abel's transformation we get that

$$\begin{aligned}s_n^\lambda(f;x)-N_n^\lambda(f;x)&=\frac{1}{P_{\lambda(n)}}\sum_{m=1}^{\lambda(n)}\Delta_m(m^{-1}(P_{\lambda(n)}-P_{\lambda(n)-m}))\\ &\quad\times\sum_{k=1}^{m}kQ_k(f;x)+\frac{1}{(\lambda(n)+1)}\sum_{k=1}^{\lambda(n)}kQ_k(f;x).\end{aligned}\tag{4}$$

Using (4) we have

$$\begin{aligned}\left\|s_n^\lambda(f)-N_n^\lambda(f)\right\|_{L^p_\omega}&\le\left|\frac{1}{P_{\lambda(n)}}\sum_{m=1}^{\lambda(n)}\Delta_m(m^{-1}(P_{\lambda(n)}-P_{\lambda(n)-m}))\right|\\ &\quad\times\left\|\sum_{k=1}^{m}kQ_k(f)\right\|_{L^p_\omega}+\frac{1}{(\lambda(n)+1)}\left\|\sum_{k=1}^{\lambda(n)}kQ_k(f;x)\right\|_{L^p_\omega}.\end{aligned}\tag{5}$$

It is clear that

$$s_n(f,x)-\sigma_n(f;x)=\frac{1}{n+1}\sum_{k=1}^{n}kQ_k(f;x).$$

Then by Lemma 2 and (6) we get

$$\left\|\sum_{k=1}^{n}kQ_k(f)\right\|_{L^p_\omega}=(n+1)\left\|s_n(f)-\sigma_n(f)\right\|=O(1).\tag{7}$$

Thus (5) and (7) yield

$$\begin{aligned}\left\| s_n^{\lambda}(f) - N_n^{\lambda}(f) \right\|_{L_\omega^p} &= O(\frac{1}{P_{\lambda(n)}}) \sum_{m=1}^{\lambda(n)} \left| \Delta_m (m^{-1}(P_{\lambda(n)} - P_{\lambda(n)-m})) \right| \\ &\quad + O((\lambda(n))^{-1}). \end{aligned} \tag{8}$$

According to [20] the following relations hold :

$$\begin{aligned}\Delta_m(m^{-1}(P_{\lambda(n)} - P_{\lambda(n)-m}) = \\ &= \frac{1}{m}\Delta_m(P_{\lambda(n)} - P_{\lambda(n)-m}) + \frac{P_{\lambda(n)} - P_{\lambda(n)-m-1}}{m(m+1)} \\ &= \frac{P_{\lambda(n)-m-1} - P_{\lambda(n)-m}}{m} + \frac{P_{\lambda(n)} - P_{\lambda(n)-m-1}}{m(m+1)} \\ &= \frac{P_{\lambda(n)} - P_{\lambda(n)-m-1}}{m(m+1)} - \frac{p_{\lambda(n)-m}}{m} \\ &= \frac{1}{m(m+1)} \left[P_{\lambda(n)} - P_{\lambda(n)-m-1} - (m+1)p_{\lambda(n)-m} \right]. \end{aligned} \tag{9}$$

$$\Delta_m(\frac{P_{\lambda(n)} - P_{\lambda(n)-m}}{m}) = \frac{1}{m(m+1)} \times \left[\sum_{k=\lambda(n)-m}^{\lambda(n)} p_k - (m+1)p_{\lambda(n)-m} \right] \tag{10}$$

Next we will prove by the induction the inequality

$$\left| \sum_{k=\lambda(n)-m}^{\lambda(n)} p_k - (m+1)p_{\lambda(n)-m} \right| \le \sum_{k=1}^{m} k \left| p_{\lambda(n)-k+1} - p_{\lambda(n)-k} \right|. \tag{11}$$

Let $m = 1$. Then we obtain

$$\left| \sum_{k=\lambda(n)-1}^{\lambda(n)} p_k - 2p_{\lambda(n)-1} \right| = \left| p_{\lambda(n)} - p_{\lambda(n)-1} \right|.$$

That is, the inequality (11) holds, for $m = 1$. Now we suppose that the inequality (11) holds for $m = j$. We prove the inequality for $m = j + 1$ $(\le \lambda(n))$.
The estimate

$$\begin{aligned} \left| \sum_{k=\lambda(n)-(j+1)}^{\lambda(n)} p_k - (j+2)p_{\lambda(n)-(j+1)} \right| &= \left| \sum_{k=\lambda(n)-j}^{\lambda(n)} p_k - (j+1)p_{\lambda(n)-(j+1)} \right| \\ = \left| \sum_{k=\lambda(n)-j}^{\lambda(n)} p_k - (j+1)p_{\lambda(n)-j} + (j+1)p_{\lambda(n)-j} - (j+1)p_{\lambda(n)-(j+1)} \right| \\ \le \left| \sum_{k=\lambda(n)-j}^{\lambda(n)} p_k - (j+1)p_{\lambda(n)-1} \right| + \left| (j+1)p_{\lambda(n)-j} - (j+1)p_{\lambda(n)-(j+1)} \right| \end{aligned}$$

$$\leq \sum_{k=1}^{j} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right|+(j+1)\left|p_{\lambda(n)-j}-p_{\lambda(n)-(j+1)}\right|$$
$$=\sum_{k=1}^{j+1} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right|.$$

holds. That is, (11) is true for $m=j+1$. Thus the inequality (11) is proved for any $1\leq m\leq\lambda(n)$. By (10) and (11) we have

$$\begin{aligned}
&\sum_{m=1}^{\lambda(n)}\left|\Delta_m\left(\frac{P_{\lambda(n)}-P_{\lambda(n)-m}}{m}\right)\right| \\
&\leq \sum_{m=1}^{\lambda(n)}\frac{1}{m\,(m+1)}\sum_{k=1}^{m} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right| \\
&\leq \sum_{k=1}^{\lambda(n)} k\left|p_{\lambda(n)-k+1}-p_{\lambda(n)-k}\right|\sum_{m=k}^{\infty}\frac{1}{m\,(m+1)} \qquad (12)\\
&=\sum_{k=0}^{\lambda(n)-1}\left|\Delta p_k\right|.
\end{aligned}$$

According to condition of the theorem 1 the relation

$$\sum_{k=0}^{\lambda(n)-1}\left|\Delta p_k\right|=O(P_{\lambda(n)}/\lambda(n)) \tag{13}$$

holds. Then use of (12), (13) and (8) gives us

$$\left\|s_n^{\lambda}(f)-N_n^{\lambda}(f)\right\|_{L_\omega^p}=O((\lambda(n))^{-1}).$$

Thus from (14) and Lemma 1 for $\alpha=1$ we obtain

$$\left\|f-N_n^{\lambda}(f)\right\|_{L_\omega^p}=O((\lambda(n))^{-1}).$$

Case (iii): First of all we prove the estimate

$$\sum_{m=1}^{\lambda(n)}\Delta_m\left(\frac{P_{\lambda(n)}-P_{\lambda(n)-m}}{m}\right)=O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right). \tag{14}$$

According to condition in the case (iii) of theorem 1 the estimate

$$\sum_{k=1}^{\lambda(n)-1} k\left|\Delta p_k\right|=O(P_{\lambda(n)}) \tag{15}$$

holds. Using (10) and (11) we have

$$\begin{aligned}
&\sum_{m=1}^{\lambda(n)} \Delta_m \left(\frac{P_{\lambda(n)} - P_{\lambda(n)-m}}{m} \right) \\
&\leq \sum_{m=1}^{\lambda(n)} \frac{1}{m\,(m+1)} \sum_{k=1}^{m} k \left| \Delta_k p_{\lambda(n)-k} \right| \\
&= \sum_{m=1}^{r} \frac{1}{m\,(m+1)} \sum_{k=1}^{m} k \left| \Delta_k p_{\lambda(n)-k} \right| + \sum_{m=r+1}^{\lambda(n)} \frac{1}{m\,(m+1)} \sum_{k=1}^{m} k \left| \Delta_k p_{\lambda(n)-k} \right| \\
&:= S_1 + S_2 \qquad (16)
\end{aligned}$$

Let r denote the integral part of $(\lambda(n)/2)$. Then Abel's transformation and using (15), we get that

$$\begin{aligned}
S_1 &= \sum_{m=1}^{r} \frac{1}{m\,(m+1)} \sum_{k=1}^{m} k \left| \Delta_k p_{\lambda(n)-k} \right| \\
&\leq \sum_{k=1}^{r} \left| \Delta_k p_{\lambda(n)-k} \right| \leq \sum_{j=r-2}^{\lambda(n)-1} \left| \Delta p_j \right| = O\left(\frac{P_{\lambda(n)}}{\lambda(n)} \right). \qquad (17)
\end{aligned}$$

For S_2 we can write the following:

$$\begin{aligned}
S_2 &= \sum_{m=r+1}^{\lambda(n)} \frac{1}{m\,(m+1)} \sum_{k=1}^{m} k \left| \Delta p_{\lambda(n)-k} \right| \\
&= \sum_{m=r+1}^{\lambda(n)} \frac{1}{m\,(m+1)} \sum_{k=1}^{r} k \left| \Delta p_{\lambda(n)-k} \right| + \sum_{m=r+1}^{\lambda(n)} \frac{1}{m\,(m+1)} \sum_{k=r}^{m} k \left| \Delta p_{\lambda(n)-k} \right| \\
&:= S_{21} + S_{22}. \qquad (18)
\end{aligned}$$

If using again the condition (15) we have

$$\begin{aligned}
S_{21} &\leq \sum_{m=r}^{\lambda(n)} \frac{1}{(m+1)} \sum_{j=r-2}^{\lambda(n)-1} \left| \Delta p_j \right| = O\left(\frac{P_{\lambda(n)}}{\lambda(n)} \right) \\
S_{22} &\leq \sum_{m=r}^{\lambda(n)} \frac{1}{(m+1)} \sum_{k=r}^{m} \left| \Delta p_{\lambda(n)-k} \right| \\
&= O\left(\frac{1}{\lambda(n)} \right) \left[\left| \Delta p_0 \right| + 2 \left| \Delta p_1 \right| + ... + (r+1) \left| \Delta p_{r+1} \right| \right] \\
&= O\left(\frac{P_{\lambda(n)}}{\lambda(n)} \right). \qquad (19)
\end{aligned}$$

Now combining (16)-(19) we obtain (14). From (8), (14) and Lemma 1 we get

$$\begin{aligned}\left\|f-N_n^{\lambda}(f)\right\|_{L_\omega^p} &= \left\|f-s_n^{\lambda}(f)+s_n^{\lambda}(f)-N_n^{\lambda}(f)\right\|_{L_\omega^p}\\ &\leq \left\|f-s_n^{\lambda}(f)\right\|_{L_\omega^p}+\left\|s_n^{\lambda}(f)-N_n^{\lambda}(f)\right\|_{L_\omega^p}\leq O((\lambda(n))^{-1}).\end{aligned}$$

and the proof is completed.

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