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QUASI-ASYMPTOTIC BEHAVIOR AT THE ORIGIN OF TEMPERED OPERATORS

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Abstract. A subspace of Mikusiński operators, which was introduced by K. Yosida, is used to investigate quasi-asymptotic behavior at the origin in one dimension. As an application, an Abelian theorem of the initial type for the Stieltjes transform is established.

1. Introduction

Asymptotic behavior of generalized functions is an active area of research with applications in differential equations, integral transforms, and quantum physics.

Quasi-asymptotic behavior of Schwartz distributions was introduced in the early 1970s by Zav'yalov [11] and investigated by Vladimirov, Drozhzhinov and Zav'yalov (see [9] and references in [7]). More recently, Pilipović, Stanković, Vindas and others have continued the investigation. For an excellent account of quasi-asymptotic behavior of distributions the reader is referred to [7].

By using a subspace of Mikusiński operators introduced by K. Yosida [10], an elementary approach to quasi-asymptotic behavior at infinity has been investigated [4]. The approach is algebraic and is elementary in that it requires some basic concepts from analysis, but none from functional analysis such as duality theory as do other approaches. In this note, using the same construction, quasi-asymptotic behavior at the origin is studied. As an application, an Abelian theorem of the initial type for the Stieltjes transform is established.

2. The space of tempered operators

In this section, a brief summary of the needed preliminaries is given. For more details, the reader is referred to [4].

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Let $C_+(\mathbb{R})$ denote the space of all continuous functions on $\mathbb{R}$ which vanish on the interval $(-\infty, 0)$.

For $f, g \in C_+(\mathbb{R})$, the convolution is given by

$$(f * g)(x) = \int_0^x f(x-t)g(t)\,dt.$$

The space of tempered functions supported on $[0,\infty)$ will be denoted by $C_+^t(\mathbb{R})$. That is, $f \in C_+^t(\mathbb{R})$ provided $f \in C_+(\mathbb{R})$ and there exists $m \in \mathbb{N}$ such that $f(x)x^{-m}$ is bounded as $x \to \infty$.

Notice that for $f, g \in C_+^t(\mathbb{R})$, then $f+g, f*g \in C_+^t(\mathbb{R})$.

Let H denote the Heaviside function. That is, $H(x) = 1$ for $x \geq 0$ and zero otherwise. For each $n \in \mathbb{N}$, we denote by H^n the function $H * \ldots * H$, where H is repeated n times.

For $k = 1, 2, \ldots$

$$\mathcal{M}_k = \left\{ \frac{f}{H^k} \ : \ f \in C_+^t(\mathbb{R}) \right\}.$$

It should be noted that, in the definition of $\mathcal{M}_k$, the quotients $\frac{f}{H^k}$ are convolution quotients.

Definition 1. Let $k \in \mathbb{N}$ and $W_n, W \in \mathcal{M}_k$, $n \in \mathbb{N}$. The sequence $\{W_n\}$ converges to W in $\mathcal{M}_k$, denoted $W_n \to W$ in $\mathcal{M}_k$, provided $W_n = \frac{f_n}{H^k}$ and $W = \frac{f}{H^k}$ such that for some $m \in \mathbb{N}$,

$$\sup_{x \geq 0} \left| \frac{f_n(x) - f(x)}{1 + x^m} \right| \to 0 \text{ as } n \to \infty.$$

The space of tempered operators $\mathcal{M}^\tau$ is defined as a countable union space. That is,

$$\mathcal{M}^\tau = \bigcup_{k=1}^{\infty} \mathcal{M}_k.$$

Two elements of $\mathcal{M}^\tau$ are equal, denoted $\frac{f}{H^n} = \frac{g}{H^m}$, if and only if $H^m * f = H^n * g$.

Clearly, $\mathcal{M}_k \subset \mathcal{M}_{k+1}$, $k \in \mathbb{N}$. Moreover, if $W_n, W \in \mathcal{M}_k$ ($n \in \mathbb{N}$ and some $k \in \mathbb{N}$) such that $W_n \to W$ in $\mathcal{M}_k$, then $W_n \to W$ in $\mathcal{M}_{k+1}$.

Definition 2. Let $W_n, W \in \mathcal{M}^\tau$, $n \in \mathbb{N}$. The sequence $\{W_n\}$ converges to W in M^τ, denoted $\mathcal{M}^\tau\text{-}\lim_{n\to\infty} W_n = W$, provided there exists $k \in \mathbb{N}$ such that $W_n, W \in \mathcal{M}_k$ for all $n \in \mathbb{N}$, and $W_n \to W$ in $\mathcal{M}_k$.

Let $\alpha \in \mathbb{C}$ and $\frac{f}{H^k}, \frac{g}{H^n} \in \mathcal{M}^\tau$. Then with scalar multiplication, addition, and multiplication (convolution) as follows, $\mathcal{M}^\tau$ is an algebra with identity $\delta = \frac{H^2}{H^2}$.

(1) $\alpha \frac{f}{H^k} = \frac{\alpha f}{H^k}$

(2) $\frac{f}{H^k} + \frac{g}{H^n} = \frac{H^n * f + H^k * g}{H^{k+n}}$
(3) $\frac{f}{H^k} * \frac{g}{H^n} = \frac{f*g}{H^{k+n}}$

Let $f \in C_+^t(\mathbb{R})$. Then, $W^f = \frac{H*f}{H} \in \mathcal{M}^\tau$. Therefore, $C_+^t(\mathbb{R})$ can be identified with a subspace of $\mathcal{M}^\tau$.

3. Quasi-Asymptotic Behavior at the Origin

A real-valued function $L(x)$ is slowly varying at the origin [8], if it is positive, measurable on $(0,a)$ for some $a>0$, and such that for each $\varepsilon > 0$,

$$\lim_{x\to 0^+} \frac{L(\varepsilon x)}{L(x)} = 1.$$

Unless otherwise stated $\alpha \in \mathbb{R}$ and L will denote a slowly varying function.

Let $W = \frac{f}{H^k} \in \mathcal{M}_k$. For $\varepsilon > 0$, define $W(\varepsilon x) = \frac{1}{\varepsilon^k} \frac{f(\varepsilon x)}{H^k}$ $\left(W_\varepsilon = \frac{1}{\varepsilon^k} \frac{f_\varepsilon}{H^k} \right)$.

Definition 3. Let $W \in \mathcal{M}^\tau$. Then, W is said to have *quasi-asymptotic behavior at the origin related to* $\varepsilon^\alpha L(\varepsilon)$ provided there exists $V \in \mathcal{M}^\tau$ such that

$$\mathcal{M}^\tau\text{-} \lim_{\varepsilon \to 0^+} \frac{W(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)} = V.$$

This will be denoted $W \overset{q}{\sim} V$ at 0^+ related to $\varepsilon^\alpha L(\varepsilon)$.

The generalized derivative and multiplication by "x" are defined as follows. Let $W = \frac{f}{H^k} \in \mathcal{M}^\tau$. Then,

(i) $W^{(n)} = \frac{f}{H^{k+n}}$, for $n = 1, 2, \ldots$
(ii) $xW = \frac{xf - kH*f}{H^k}$ $(k \geq 2)$ and $x^n W = x(x^{n-1}W)$, $n = 2, 3, \ldots$.

Theorem 4. *Let $n \in \mathbb{N}$ and $W, V \in \mathcal{M}^\tau$. If $W \overset{q}{\sim} V$ at 0^+ related to $\varepsilon^\alpha L(\varepsilon)$, then*

(i) *$W^{(n)} \overset{q}{\sim} V^{(n)}$ at 0^+ related to $\varepsilon^{\alpha - n} L(\varepsilon)$.*
(ii) *$x^n W \overset{q}{\sim} x^n V$ at 0^+ related to $\varepsilon^{\alpha+n} L(\varepsilon)$.*

The proof of (i) follows directly from the definition. For the proof of (ii), replace λ with ε and $\lambda \to \infty$ with $\varepsilon \to 0^+$ in the proof of Proposition 3.2 in [4].

Definition 5. A tempered operator $W = \frac{f}{H^k}$ is said to *vanish on an open interval* (a,b), denoted $W(x) = 0$ on (a,b), provided there exists a polynomial p with degree at most $k-1$ such that $f(x) = p(x)$ for $a < x < b$.

Let $W, V \in \mathcal{M}^\tau$. W and V are said to be equal on an interval (a,b) provided $W - V$ vanishes on (a,b).

The next theorem shows that quasi-asymptotic behavior at the origin is a local property. This is not necessarily the case for quasi-asymptotic behavior at infinity unless $\alpha > -1$ [4].

Theorem 6. *Let* $W, V, U \in \mathcal{M}^\tau$ *such that* W *and* V *are equal in some neighborhood of the origin. If* $W \overset{q}{\sim} U$ *at* 0^+ *related to* $\varepsilon^\alpha L(\varepsilon)$*, then* $V \overset{q}{\sim} U$ *at* 0^+ *related to* $\varepsilon^\alpha L(\varepsilon)$*.*

Proof. Notice that

$$\frac{V_\varepsilon}{\varepsilon^\alpha L(\varepsilon)} = \frac{(V-W)_\varepsilon}{\varepsilon^\alpha L(\varepsilon)} + \frac{W_\varepsilon}{\varepsilon^\alpha L(\varepsilon)}. \tag{3.1}$$

The conclusion now follows, provided

$$\mathcal{M}^\tau\text{-}\lim_{\varepsilon\to 0^+} \frac{(V-W)_\varepsilon}{\varepsilon^\alpha L(\varepsilon)} = 0.$$

Now, $(V-W)(x) = 0$ on $(-b, b)$ for some $b > 0$. Thus, $V - W = \frac{f}{H^k}$, where $k \in \mathbb{N}$ and $f \in C^t_+(\mathbb{R})$ such that $f(x) = p(x)$ on $(-b, b)$ for some polynomial p with degree at most $k-1$. However, $f(x) = 0$ on $(-b, 0)$, and thus $f(x) = 0$ on $(-b, b)$.

Therefore, for some $M > 0$, $|f(x)| \le Mx^n$, $x \ge 0$. We can take $n \in \mathbb{N}$ such that $n > k + \alpha$.

Now,

$$\frac{(V-W)_\varepsilon}{\varepsilon^\alpha L(\varepsilon)} = \frac{1}{\varepsilon^{k+\alpha} L(\varepsilon)} \frac{f_\varepsilon}{H^k}.$$

Then, for some $\gamma > 0$, all $x \ge 0$ and $\varepsilon > 0$,

$$\left|(1+x^n)^{-1} \frac{f(\varepsilon x)}{\varepsilon^{k+\alpha} L(\varepsilon)}\right| \le \frac{M}{(1/\varepsilon)^\gamma L^*(1/\varepsilon)}\,, \tag{3.2}$$

where L^* is the slowly varying function at infinity $L^*(x) = L\left(\frac{1}{x}\right)$.

Since $\lim_{\varepsilon\to 0^+} (1/\varepsilon)^\gamma L^*(1/\varepsilon) = \infty$ [8], (3.2) yields

$$\mathcal{M}^\tau\text{-}\lim_{\varepsilon\to 0^+} \frac{(V-W)_\varepsilon}{\varepsilon^\alpha L(\varepsilon)} = 0.$$

This completes the proof. □

Let g be a locally integrable function (i.e. $g \in L^1_{loc}(\mathbb{R})$) such that $\operatorname{supp} g \subseteq [0, \infty)$ and $g(x) = O(x^m)$ as $x \to \infty$, for some $m \in \mathbb{N}$. Then g can be identified with $W^g = \frac{H * g}{H} \in \mathcal{M}^\tau$.

For the next theorem, $n \in \mathbb{N} \cup \{0\}$, $L_n(x) = (-1)^n \ln^n x$ for $0 < x < 1$ and $L_n(x) = 1$ for $x \ge 1$. (Note. $\ln^n x = \ln x \cdot \ln x \cdot \ldots \cdot \ln x$ (n times).)

Theorem 7. *Let* $f \in L^1_{loc}(\mathbb{R})$ *such that* $\operatorname{supp} f \subseteq [0, \infty)$ *and* $f(x) = O(x^m)$ *as* $x \to \infty$*, for some* $m \in \mathbb{N}$*. If* $\lim\limits_{x\to 0^+} \dfrac{f(x)}{x^\alpha L_n(x)} = \gamma$*, for some* $\alpha > -1$, $\gamma \in \mathbb{C}$*, and* $n \in \mathbb{N} \cup \{0\}$*, then* $W^f \overset{q}{\sim} \gamma W^{g_\alpha}$ *at* 0^+ *related to* $\varepsilon^\alpha L_n(\varepsilon)$*, where* $g_\alpha(x) = H(x)x^\alpha$*.*

Proof. Let $0 < A < 1$ such that $\left|\frac{f(x)}{x^\alpha L_n(x)}\right| < |\gamma| + 1$ for $0 < x < A$. Let $B > 0$ such that $0 < A < B < 1$.

Define $g \in L^1_{loc}(\mathbb{R})$ as follows.

$$g(x) = \begin{cases} f(x) & 0 < x < A \\ \frac{1}{A-B}\int_0^A f & A \le x \le B \\ 0 & x > B. \end{cases}$$

Since quasi-asymptotic behavior at the origin is a local property (Theorem 6), it suffices to show that $W^g \overset{q}{\sim} \gamma W^{g_\alpha}$ at 0^+ related to $\varepsilon^\alpha L_n(\varepsilon)$.

We need to show that for some $m \in \mathbb{N}$,
$(1 + x^m)^{-1}\left(\int_0^x \frac{g(\varepsilon t)}{\varepsilon^\alpha L_n(\varepsilon)}\,dt - \gamma\int_0^x t^\alpha dt\right) \to 0$ uniformly on $[0, \infty)$ as $\varepsilon \to 0^+$.

This will be accomplished by showing the following.

(I) For any $b > 0$, $\int_0^x \frac{g(\varepsilon t)}{\varepsilon^\alpha L_n(\varepsilon)}\,dt \to \gamma\int_0^x t^\alpha dt$ uniformly on $[0, b]$ as $\varepsilon \to 0^+$.

(II) There exist $M > 0$, $\varepsilon_0 > 0$, and $m \in \mathbb{N}$ such that $\left|\int_0^x \frac{g(\varepsilon t)}{\varepsilon^\alpha L_n(\varepsilon)}\,dt\right| \le Mx^m$, for all $x \ge 0$ and $0 < \varepsilon < \varepsilon_0$.

Now, let $\nu > 0$ such that $\alpha - \nu > -1$.

Thus, there exist $a > 0$ and $\varepsilon_0 > 0$ such that

$$\frac{L_n(\varepsilon t)}{L_n(\varepsilon)} < t^{-\nu}, \text{ for } 0 < t < a,\ 0 < \varepsilon < \varepsilon_0. \tag{3.3}$$

Now to show (I). First consider $0 \le x \le a$. Then,

$$\left|\int_0^x \frac{g(\varepsilon t)}{\varepsilon^\alpha L_n(\varepsilon)}\,dt - \gamma\int_0^x t^\alpha dt\right| \le \int_0^a t^\alpha \left|\frac{g(\varepsilon t)}{(\varepsilon t)^\alpha L_n(\varepsilon t)}\,\frac{L_n(\varepsilon t)}{L_n(\varepsilon)} - \gamma\right| dt. \tag{3.4}$$

Thus, by (3.3), (3.4), and $\frac{g(\varepsilon a)}{(\varepsilon a)^\alpha L_n(\varepsilon a)} \to \gamma$ as $\varepsilon \to 0^+$, it follows that

$$\int_0^x \frac{g(\varepsilon t)}{\varepsilon^\alpha L_n(\varepsilon)}\,dt \to \gamma\int_0^x t^\alpha dt \text{ uniformly on } [0, a] \text{ as } \varepsilon \to 0^+.$$

Now consider $0 \le x \le b$, where $b > a$. Then,

$$\left|\int_0^x \frac{g(\varepsilon t)}{\varepsilon^\alpha L_n(\varepsilon)}\,dt - \gamma\int_0^x t^\alpha dt\right| \le \int_0^b t^\alpha \left|\frac{g(\varepsilon t)}{(\varepsilon t)^\alpha L_n(\varepsilon t)}\,\frac{L_n(\varepsilon t)}{L_n(\varepsilon)} - \gamma\right| dt$$

$$= \int_0^a t^\alpha \left|\frac{g(\varepsilon t)}{(\varepsilon t)^\alpha L_n(\varepsilon t)}\,\frac{L_n(\varepsilon t)}{L_n(\varepsilon)} - \gamma\right| dt + \int_a^b t^\alpha \left|\frac{g(\varepsilon t)}{(\varepsilon t)^\alpha L_n(\varepsilon t)}\,\frac{L_n(\varepsilon t)}{L_n(\varepsilon)} - \gamma\right| dt.$$

By above, we only need to show that

$$\int_a^b t^\alpha \left|\frac{g(\varepsilon t)}{(\varepsilon t)^\alpha L_n(\varepsilon t)}\,\frac{L_n(\varepsilon t)}{L_n(\varepsilon)} - \gamma\right| dt \to 0 \text{ as } \varepsilon \to 0^+.$$

However, this follows from $\frac{g(\varepsilon t)}{(\varepsilon t)^\alpha L_n(\varepsilon t)} \to \gamma$ uniformly on $[a,b]$ as $\varepsilon \to 0^+$, and $\frac{L_n(\varepsilon t)}{L_n(\varepsilon)} \to 1$ uniformly on $[a,b]$ as $\varepsilon \to 0^+$. Therefore,

$$\int_0^x \frac{g(\varepsilon t)}{\varepsilon^\alpha L_n(\varepsilon)}\,dt \to \gamma \int_0^x t^\alpha dt \text{ uniformly on } [0,b] \text{ as } \varepsilon \to 0^+.$$

Thus, (I) is verified.

To verify (II), we consider three cases.

Case 1. $\varepsilon x \geq B$.

Then, $\frac{(H*g)(\varepsilon x)}{\varepsilon^{\alpha+1} L_n(\varepsilon)} = 0$.

Case 2. $0 < \varepsilon x < A \quad (0 < \varepsilon < A)$.

Now, since $\left|\frac{g(x)}{x^\alpha L_n(x)}\right| < |\gamma| + 1$, it follows that $\frac{(H*g)(\varepsilon x)}{\varepsilon^{\alpha+1} L_n(\varepsilon)}$ is bounded on $0 < \varepsilon x < A, 0 < \varepsilon < A$.

Thus,

$$\begin{aligned}\left|\frac{(H*g)(\varepsilon x)}{\varepsilon^{\alpha+1} L_n(\varepsilon)}\right| &= \left|\frac{x^{\alpha+1}(H*g)(\varepsilon x)}{(\varepsilon x)^{\alpha+1} L_n(\varepsilon x)}\,\frac{L_n(\varepsilon x)}{L_n(\varepsilon)}\right| \\ &\leq M_1 x^{\frac{\alpha+1}{2}}\left(x^{\frac{\alpha+1}{2}} + \frac{x^{\frac{\alpha+1}{2}} L_n(x)}{L_n(A)}\right) \\ &\leq M_2 x^{\alpha+1}\end{aligned}$$

(where M_1 and M_2 are positive constants that are independent of both x and ε).

Case 3. $A \leq \varepsilon x \leq B \quad (0 < \varepsilon < A)$.

Since $g \in L^1_{loc}(\mathbb{R})$, $|(H*g)(\varepsilon x)| \leq \int_0^B |g|$, for $A \leq \varepsilon x \leq B$.

By using the definition of L_n and the above, it follows that there exists $M_3 > 0$ such that

$$\left|\frac{(H*g)(\varepsilon x)}{\varepsilon^{\alpha+1} L_n(\varepsilon)}\right| = \left|\frac{x^{\alpha+1}(H*g)(\varepsilon x)}{(\varepsilon x)^{\alpha+1} L_n(\varepsilon x)}\,\frac{L_n(\varepsilon x)}{L_n(\varepsilon)}\right| \leq M_3 x^{\alpha+1}.$$

Thus, combining the three cases, there exist $M > 0$ and $m \in \mathbb{N}$ such that for all $x \geq 0$ and $0 < \varepsilon < A$,

$$\left|\int_0^x \frac{g(\varepsilon t)}{\varepsilon^\alpha L_n(\varepsilon)}\,dt\right| = \left|\frac{(H*g)(\varepsilon x)}{\varepsilon^{\alpha+1} L_n(\varepsilon)}\right| \leq M x^m.$$

This completes the proof. □

Let $\alpha \in \mathbb{R}$ and L a slowly varying function at the origin. A locally integrable function f is said to be asymptotic to the function $g_\alpha(x) = x^\alpha L(x)$ at the origin if

$$\lim_{x \to 0^+} \frac{f(x)}{x^\alpha L(x)} = \gamma \quad \text{a.e. } (\gamma \neq 0).$$

The next example provides an example of a locally integrable function that has quasi-asymptotic behavior but not asymptotic behavior at the origin.

Example 8. (See [6].) Let $f(x) = \begin{cases} 2+\sin\frac{1}{x} & \text{for } x>0 \\ 0 & \text{for } x<0. \end{cases}$

Since $\frac{W^f_\varepsilon}{\varepsilon^0} = \frac{1}{\varepsilon}\frac{(H*f)_\varepsilon}{H}$ and $\frac{1}{\varepsilon}(H*f)_\varepsilon(x) = \frac{1}{\varepsilon}\int_0^{\varepsilon x}\left(2+\sin\frac{1}{t}\right)dt$, it will follow that $W_f \overset{q}{\sim} 2W^H$ at 0^+ related to ε^0, provided that

$\frac{1}{\varepsilon}(1+x^m)^{-1}\int_0^{\varepsilon x}\sin\frac{1}{t}dt \to 0$ uniformly on $[0,\infty)$ as $\varepsilon\to 0^+$, for some $m\in\mathbb{N}$.

Now, $\left|\frac{1}{\varepsilon}\int_0^{\varepsilon x}\sin\frac{1}{t}dt\right| \le \frac{1}{\varepsilon}\int_0^{\varepsilon x}\left|\sin\frac{1}{t}\right|dt \le x$, for all $x\ge 0$ and $\varepsilon>0$, and thus, by taking $m=2$ it is enough to show that

$$\frac{1}{\varepsilon}\int_0^{\varepsilon x}\sin\frac{1}{t}\,dt \to 0 \text{ uniformly on compact subsets as } \varepsilon\to 0^+.$$

Thus it suffices to show that

$$\frac{1}{x}\int_0^{x}\sin\frac{1}{t}\,dt \to 0 \text{ as } x\to 0^+.$$

Using that $\int_0^x \sin\frac{1}{t}\,dt = x^2\cos\frac{1}{x} - 2\int_0^x t\cos\frac{1}{t}\,dt, \quad x>0$, we see that

$$\frac{1}{x}\int_0^{x}\sin\frac{1}{t}\,dt = x\cos\frac{1}{x} - \frac{2}{x}\int_0^x t\cos\frac{1}{t}\,dt \to 0 \text{ as } x\to 0^+.$$

Therefore,

$$W^f \overset{q}{\sim} 2W^H \text{ at } 0^+ \text{ related to } \varepsilon^0.$$

4. The Stieltjes Transform

In this section, as an application, an Abelian theorem of the initial type for the Stieltjes transform is given.

Let $r>-1$ and $f\in L^1_{loc}(\mathbb{R})$ such that supp$f\subseteq[0,\infty)$ and $f(x)x^{-r+\eta}$ is bounded as $x\to\infty$ for some $\eta>0$. Then, the Stieltjes transform of index r of f is given by

$$S^r_z f = \int_0^\infty \frac{f(x)}{(x+z)^{r+1}}\,dx,\ z\in\mathbb{C}\backslash(-\infty,0].$$

For $r\in\mathbb{R}$ and $k=1,2,\dots$

$$\mathcal{M}_k(r) = \left\{\frac{f}{H^k}\ :\ f\in C_+(\mathbb{R}), f(x)x^{-r-k+\eta} \text{ is bdd as } x\to\infty \text{ for some } \eta>0\right\}$$

and

$$\mathcal{M}(r) = \bigcup_{k=1}^\infty \mathcal{M}_k(r).$$

Let $r>-1$ and $W\in\mathcal{M}(r)$. That is, $W=\frac{f}{H^k}\in\mathcal{M}_k(r)$, for some $k\in\mathbb{N}$. The Stieltjes transform of index r of W is defined by

$$\Lambda^r_z W = (r+1)_k\int_0^\infty \frac{f(x)}{(x+z)^{r+k+1}}\,dx,\ \ z\in\mathbb{C}\backslash(-\infty,0]\,,$$

where $(r+1)_k = \frac{\Gamma(r+k+1)}{\Gamma(r+1)} = (r+1)(r+2)\dots(r+k)$ and Γ is the gamma function.

Notice that $\Lambda_z^r W = (r+1)_k S_z^{r+k} f$, where $W = \frac{f}{H^k}$.

Properties 9. [5]. Let $W = \frac{f}{H^k} \in \mathcal{M}(r)$. Then for $r > -1$ and $z \in \mathbb{C}\backslash(-\infty, 0]$,

1. $\Lambda_z^r W$ is an analytic function.
2. $\Lambda_z^r \tau_c W = \Lambda_{z+c}^r W$, $c > 0$ and $\tau_c W = \frac{\tau_c f}{H^k}$, $\tau_c f(x) = f(x-c)$.
3. $\Lambda_z^r W^{(m)} = (r+1)_m \Lambda_z^{r+m} W$, $m = 1, 2, \dots$
4. $\frac{d^m}{dz^m}\Lambda_z^r W = (-1)^m (r+1)_m \Lambda_z^{r+m} W = (-1)^m \Lambda_z^r W^{(m)}, m = 1, 2, \dots$
5. $\Lambda_z^{r+1}(xW) = \Lambda_z^r W - z\Lambda_z^{r+1} W$.

For $\sigma > 0$, let $\ell_\sigma(x) = H(x)x^\sigma$. Then, for $\alpha > -1$, let $\Theta_{\alpha+1} = \frac{1}{\Gamma(\alpha+2)}\frac{\ell_{\alpha+1}}{H}$, and for $\alpha \le -1$ and $\alpha + n > 0$, $\Theta_{\alpha+1} = \frac{1}{\Gamma(\alpha+n+1)}\frac{\ell_{\alpha+n}}{H^n}$.

Notice that, for $n = 0, 1, 2, \dots$, $\Theta_{-n} = \delta^{(n)}$. In particular, $\Theta_0 = \delta$ and $\Theta_{-1} = \delta'$. Thus, for $0 < \beta < 1$, $\Theta_{-\beta}$ can be interpreted as the fractional differential operator of order β.

The following is an Abelian theorem of the initial type.

Theorem 10. *Let* $\alpha \in \mathbb{R}$ *and* $r > -1$, $r > \alpha$. *If* $W \in \mathcal{M}(r)$ *and* $W \overset{q}{\sim} \zeta\Theta_{\alpha+1}$ $(\zeta \in \mathbb{C}, \zeta \neq 0)$ *at* 0^+ *related to* $\varepsilon^\alpha L(\varepsilon)$, *then*

$$\lim_{\substack{\varepsilon\to 0^+ \\ z\in\Omega_\psi}} \frac{\Gamma(r+1)(\varepsilon z)^{r-\alpha}\Lambda_{\varepsilon z}^r W}{\Gamma(r-\alpha)L(\varepsilon)} = \zeta,$$

where $\Omega_\psi = \{z \in \mathbb{C} : |\arg z| \le \psi < \pi\}$.

Proof. Let $n, m \in \mathbb{N}$ such that $\alpha + n > 0$, $W = \frac{f}{H^n} \in \mathcal{M}_n(r)$ and

$$\sup_{x\ge 0}(1+x^m)^{-1}\left|\frac{f(\varepsilon x)}{\varepsilon^{\alpha+n}L(\varepsilon)} - \frac{\xi}{\Gamma(\alpha+n+1)}\ell_{\alpha+n}(x)\right| \to 0 \text{ as } \varepsilon \to 0^+. \tag{4.1}$$

Since $W \in \mathcal{M}_n(r)$, $f \in \mathcal{S}'_+(\mathbb{R})$ (the space of tempered distributions supported on $[0, \infty)$). By (4.1), it follows that

$$\frac{f(\varepsilon x)}{\varepsilon^{\alpha+n}L(\varepsilon)} - \frac{\xi}{\Gamma(\alpha+n+1)}\,\ell_{\alpha+n}(x) \to 0 \text{ in } \mathcal{S}'(\mathbb{R}) \text{ as } \varepsilon \to 0^+.$$

Thus,

$$f \overset{q}{\sim} \frac{\xi}{\Gamma(\alpha+n+1)}\,\ell_{\alpha+n} \quad \text{(in } \mathcal{S}'(\mathbb{R})\text{) at } 0^+ \text{ related to } \varepsilon^{\alpha+n}L(\varepsilon).$$

By using the Abelian theorem for $\mathcal{S}'_+(\mathbb{R})$ ([6], Theorem 4.2) with $\alpha + n$ for b and $r + n$ for r, it follows that

$$\lim_{\substack{\varepsilon\to 0^+ \\ z\in\Omega_\psi}} \frac{\Gamma(r+n+1)(\varepsilon z)^{r-\alpha} S_{\varepsilon z}^{r+n} f}{\Gamma(r-\alpha)L(\varepsilon)} = \zeta. \tag{4.2}$$

The result follows by observing

$$\Lambda^r_{\varepsilon z} W = (r+1)_n S^{r+n}_{\varepsilon z} f$$

□

If $L = 1$, it follows that the convergence in (4.2) is uniform in the region Ω_ψ (see [6], Theorem 4.2), and hence, we obtain the following.

Theorem 11. *Let* $\alpha \in \mathbb{R}$ *and* $r > -1$, $r > \alpha$. *If* $W \in \mathcal{M}(r)$ *and* $W \overset{q}{\sim} \zeta \Theta_{\alpha+1} \quad (\zeta \in \mathbb{C},\, \zeta \neq 0)$ *at* 0^+ *related to* ε^α, *then*

$$\lim_{\substack{z \to 0 \\ z \in \Omega_\psi}} \frac{\Gamma(r+1) z^{r-\alpha} \Lambda^r_z W}{\Gamma(r-\alpha)} = \zeta.$$

5. Tempered Distributions

A traditional approach to distribution theory may be found in [2]. In this section we will use a different approach to tempered distributions which was introduced in [1] in order to investigate the sequential approach to distribution theory.

Let $\mathcal{D}'(\mathbb{R})$ denote the space of distributions on $\mathbb{R}$. The *kth tempered derivative of a distribution* F is given by $D^k_t F = e^{-x^2/4} D^k (e^{x^2/4} F)$, where D^k is the kth order distributional derivative.

A distribution F is said to be *tempered* if there exists a square integrable function f such that

$$D^k_t f = F, \text{ for some } k \in \mathbb{N}.$$

The space of all tempered distributions will be denoted by $\mathcal{S}'(\mathbb{R})$.
Let $\mathcal{S}'_+(\mathbb{R})$ denote the subspace of $\mathcal{S}'(\mathbb{R})$ for which elements are supported on the interval $[0, \infty)$.

A sequence of tempered distributions (F_n) is *tempered convergent to a distribution* F, provided there exist square integrable functions f_n, f such that

$$D^k_t f_n = F_n,\, D^k_t f = F, \text{ for some } k \in \mathbb{N}, \text{ and}$$

$$\int_{-\infty}^{\infty} |f_n(x) - f(x)|^2 \, dx \to 0 \text{ as } n \to \infty.$$

Now, define the mapping $\Psi : \mathcal{M}^\tau \to \mathcal{S}'_+(\mathbb{R})$ by $\Psi(\frac{f}{H^k}) = D^k f$.

It has been shown [4] that the map Ψ is a well-defined bijection such that both Ψ and its inverse preserve the corresponding convergence structures.

Thus, the space $\mathcal{M}^\tau$ with the sequential convergence defined in Section 2 is isomorphic to the space $\mathcal{S}'_+(\mathbb{R})$ with tempered convergence.

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References

[1] P. Antosik, J. Mikusiński, and R. Sikorski, *Theory of Distributions: The Sequential Approach*, PWN-Polish Scientific Publishers, Warszawa, 1973.

[2] J. Barros-Neto, *An Introduction to the Theory of Distributions*, R.E. Krieger Publishing Company, Florida, 1981.

[3] J. Mikusiński, *Operational Calculus*, Vol. I, Second Edition, International Series of Monographs in Pure and Applied Mathematics 109, Pergamon Press, Oxford, PWN Polish Scientific Publishers, Warsaw, 1983.

[4] D. Nemzer, *Quasi-Asymptotic Behavior at Infinity of Tempered Operators*, Sarajevo J. Math. 11 (24) (2015), 181–195.

[5] D. Nemzer, *Mikusiński's Operational Calculus Approach to the Distributional Stieltjes Transform*, Adv. Math. Sci J. 2 (2013), 35–42.

[6] S. Pilipović, B. Stanković, and A. Takači, *Asymptotic Behavior and Stieltjes Transformation of Distributions*, Taubner, Leipzig, 1990.

[7] S. Pilipović, B. Stanković, and J. Vindas, *Asymptotic Behavior of Generalized Functions*, Series on Analysis, Applications and Computation – Vol. 5, World Scientific Publishing, London (2012).

[8] E. Seneta, *Regularly Varying Functions*, Springer, Berlin, 1976.

[9] V. S. Vladimirov, Yu. N. Drozhzhinov, and B. I. Zav'yalov, *Tauberian Theorems for Generalized Functions*, Kluwer Academic Publishers, Dordrecht, 1988.

[10] K. Yosida, *Operational Calculus: A Theory of Hyperfunctions*, Springer-Verlag, New York, 1984.

[11] B. I. Zav'yalov, *Automodel Asymptotics of Electromagnetic Form-Factors and the Behavior of their Fourier Transforms in the Neighborhood of the Light Cone*, Teoret. Mat. Fiz. 17 (1973), 178–188 (In Russian).

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